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Approximations of Laplace Transforms and Integrated Semigroups[1]

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Let $\{f_m: m \in N\}$ be a sequence of functions from $[0, \infty)$ to a Banach space E

the convergence of their Laplace transforms. This enables us to establish new approximation theorems for r-times integrated semigroups on E, for all $r \geqslant 0$. As a consequence, an open problem for the convergence of integrated semigroups on the whole space E, is solved in essence. Moreover, we present an application to non-homogeneous Cauchy problems. © 2000 Academic Press

1. INTRODUCTION

One of the fundamental theorems in the theory of operator semigroups and abstract Cauchy problems is the Trotter–Kato theorem, which tells us that for a sequence of C_0 semigroups $\{S_m(\cdot);\ m \in N\}$ on a Banach space E satisfying the stability condition

$$\|S_m(t)\| \leqslant Me^{\omega t}, \qquad m \in N, \quad t \geqslant 0, \tag{1.1}$$

(M, ω are constants), it converges (pointwise on E and uniformly on bounded intervals of $t \geqslant 0$) if and only if the sequence of resolvents $R(\lambda; A_m)$ converges (pointwise on E and for $\lambda > \omega$), where A_m is the generator of $S_m(t)$. Since the Trotter–Kato theorem came out in the late fifties, we have seen many interesting variants (cf., e.g., [10, 13, 15, 16, 19, 22]). Several years ago, a Laplace transform version of the Trotter–Kato theorem (LTV Trotter–Kato theorem in short) appeared in [4, 11, 17]. It says that the convergence of a sequence of functions f_m: $[0, \infty) \to E$ is

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equivalent to the convergence of their Laplace transforms $\int_0^\infty e^{-\lambda t} f_m(t)\, dt$, whenever the f_m satisfy the following local Lipschitz continuity condition:

$$f_m(0)=0, \qquad \|f_m(t+h)-f_m(t)\| \leqslant Mhe^{\omega(t+h)}, \qquad m\in N, \quad t,\, h\geqslant 0. \tag{1.2}$$

We emphasize that the constants M, ω in (1.2) are independent of m as well as t and h. Moreover, it is clear that condition (1.2) implies

$$\|f_m(t)\| \leqslant Me^{(\omega+1)\,t}, \qquad m\in N, \quad t\geqslant 0. \tag{1.3}$$

The LTV Trotter–Kato theorem was used in [17] to obtain the extensions of the Trotter–Kato theorem to r-times integrated semigroups for $r\in N$. Such extensions were also given in [5, 18] with proofs of operator theoretical nature. However, there exists a weakpoint among these theorems (by the reason that the generators of integrated semigroups, unlike C_0 semigroups, may fail to possess dense domains). That is, the convergence of integrated semigroups $S_m(\cdot)$ on the whole space E (pointwise) is obtained only under the additional condition that $S_m(\cdot)$ are locally Lipschitz continuous (with constants independent of m) besides (1.1) and the convergence of the resolvents: "Without this condition, it is an open problem, which seems to be difficult to solve" [18, p. 314].

The present paper aims at the open problem. First in Section 2, we consider a similar problem in a more general setting based on Laplace transforms. One advantage of the consideration is that a theorem for Laplace transforms produces results for many types of operator families and linear evolution equations in the same time. We develop completely the LTV Trotter–Kato theorem mentioned above, for a sequence of Banach space valued functions $\{f_m;\, m\in N\}$ fulfilling (1.3), by replacing (1.2) with the much weaker one that $\{f_m;\, m\in N\}$ *is equicontinuous at each point* $t\in[0,\infty)$ and showing that the equicontinuity is necessary for $\{f_m;\, m\in N\}$ to converge uniformly on bounded t-intervals. Moreover, we give an example to illustrate that without the equicontinuity, the convergence of $f_m(t)$ can not be guaranteed even for a single t and even in the scalar case. A typical example for the equicontinuity is the local Hölder continuity (with the related constants independent of m), which is weaker than the local Lipschitz continuity. In the proof of our theorem, we use the idea of reducing the approximation to a stationary problem, which originates from [14] and has been propagated by Goldstein (see [8, 9]). It was also used in [17] to obtain the previous LTV Trotter–Kato theorem, by invoking the integrated version (see [1]) of the classical Widder representation theorem for Laplace transforms. It can be seen that our approach is much more straightforward and concise, with the aid of the classical Post–Widder inversion formula for Laplace transforms.

This work in terms of Laplace transforms enables us to establish, in Section 3, new approximation theorems for r-times integrated semigroups for all $r \geqslant 0$, which cover the corresponding results in [5, 17, 18] (see Remark 3.9). As a consequence, we solve the above open problem in essence (see Theorem 3.3). An interesting application to nonhomogeneous Cauchy problems is shown in Section 4. Finally in Section 5, some examples are given.

Throughout this paper, E is a Banach space, A a linear operator in E, and M, ω, r are nonnegative constants. We will write $\mathbf{L}(E)$ for the space of all bounded linear operators from E to E. By $\mathscr{D}(A)$, $\mathscr{R}(A)$, $\mathscr{N}(A)$, $\rho(A)$, and $R(\lambda; A)$, we denote respectively the domain, the range, the kernel, the resolvent set, and the resolvent of A. N will be the set of positive integers and $N_0 := N \cup \{0\}$. $[r]$ denotes the least integer $> r-1$. Finally

$$(j_\alpha * g)(t) := \begin{cases} \displaystyle\int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\, g(s)\, ds & \text{if} \quad \alpha > -1, \\ g(t) & \text{if} \quad \alpha = -1. \end{cases}$$

2. APPROXIMATION OF LAPLACE TRANSFORMS

LEMMA 2.1. *For each $m \in N$ let $f_m \in L^1_{loc}([0, \infty), E)$ with*

$$\left\| \int_0^t f_m(s)\, ds \right\| \leqslant M e^{\omega t}, \qquad t \geqslant 0, \tag{2.1}$$

and let

$$F_m(\lambda) = \int_0^\infty e^{-\lambda t} f_m(t)\, dt, \qquad \lambda > \omega. \tag{2.2}$$

Assume that

$$\lim_{m \to \infty} F_m(\lambda) \text{ exists for } \lambda > \omega, \tag{2.3}$$

and that for a fixed $t_0 \in (0, \infty)$, $\sup_{m \in N} \|f_m(t_0)\| < \infty$ and

$$\lim_{h \to 0} \frac{1}{h} \int_0^h (f_m(t_0 + s) - f_m(t_0))\, ds = 0 \tag{2.4}$$

with uniform convergence for $m \in N$. Then $\lim_{m \to \infty} f_m(t_0)$ exists.

Proof. Set

$$\mathbf{E} = \{\mathbf{u} := (u_m)_{m \in N} \subset E;\ \sup_{m \in N} \|u_m\| < \infty\}.$$

It is clear that $\mathbf{E}$ is a Banach space under the norm $\|\mathbf{u}\|_{\mathbf{E}} := \sup_{m \in N} \|u_m\|$. Let

$$\mathbf{E}_0 = \{\mathbf{u} := (u_m)_{m \in N} \subset E;\ \lim_{m \to \infty} u_m \text{ exists}\}.$$

Then $\mathbf{E}_0$ is a closed subspace of $\mathbf{E}$. We define $\mathbf{F}: (\omega, \infty) \to \mathbf{E}$ by

$$\mathbf{F}(\lambda) = (F_m(\lambda))_{m \in N} \qquad (\lambda > \omega), \tag{2.5}$$

and $\mathbf{g}: [0, \infty) \to \mathbf{E}$ by

$$\mathbf{g}(t) = \left(\int_0^t (t-s)\, f_m(s)\, ds\right)_{m \in N} \qquad (t \geqslant 0).$$

It is clear by (2.1) and (2.2) that $\mathbf{g}$ is a continuous $\mathbf{E}$-valued function, $\|\mathbf{g}(t)\| \leqslant Mte^{\omega t}$ $(t \geqslant 0)$, and

$$\mathbf{F}(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda t} \mathbf{g}(t)\, dt, \qquad \lambda > \omega + 1.$$

Thus, it follows from (2.3) that $\mathbf{F}(\lambda) \in \mathbf{E}_0$ and therefore

$$\mathbf{F}^{(k)}(\lambda) \in \mathbf{E}_0, \qquad \lambda > \omega + 1, \qquad k \in N. \tag{2.6}$$

Moreover we see easily

$$\begin{aligned} &(-1)^k \frac{1}{k!} \left(\frac{k}{t_0}\right)^{k+1} \mathbf{F}^{(k)}(\lambda) \\ &= \left((-1)^k \frac{1}{k!} \left(\frac{k}{t_0}\right)^{k+1} F_m^{(k)}(\lambda)\right)_{m \in N}, \qquad \lambda > \omega + 1, k \in N. \end{aligned}$$

Examining the proof of the Post–Widder inversion formula for Laplace transforms (cf. [4, 11] or [24, Chap. 1]) and using (2.4) yields that

$$(-1)^k \frac{1}{k!} \left(\frac{k}{t_0}\right)^{k+1} F_m^{(k)}\left(\frac{k}{t_0}\right) \to f_m(t_0), \qquad \text{as} \quad k \to \infty,$$

uniformly for all $m \in N$; that is,

$$\mathbf{f}(t_0) = \lim_{k \to \infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t_0}\right)^{k+1} \mathbf{F}^{(k)}\left(\frac{k}{t_0}\right), \tag{2.7}$$

where $\mathbf{f}(t_0) := (f_m(t_0))_{m \in N}$. So, (2.6) implies that $\mathbf{f}(t_0) \in \mathbf{E}_0$, since $\mathbf{E}_0$ is closed. This finishes the proof.

THEOREM 2.2. *For each $m \in N$, let $f_m \in C([0, \infty), E)$ satisfying* (1.3) *and let F_m be as in* (2.2). *Then the following assertions are equivalent.*

(i) *$\{f_m;\ m \in N\}$ is equicontinuous at each point $t \in [0, \infty)$, and $\lim_{m \to \infty} F_m(\lambda)$ exists for $\lambda > \omega + 1$.*

(ii) *$\lim_{m \to \infty} f_m(t)$ exists for $t \geqslant 0$ and the convergence is uniform on bounded t-intervals.*

Proof. (i) ⇒ (ii). An application of Lemma 2.1 yields immediately that for each $t \in (0, \infty)$, $\lim_{m \to \infty} f_m(t)$ exists. We now fix $b > 0$. Then for each $\varepsilon > 0$, there exists $k_\varepsilon \in N$ such that

$$\|f_m(t) - f_m(s)\| < \frac{\varepsilon}{3}, \qquad m \in N,\ t, s \in [0, b] \text{ with } |t - s| \leqslant b k_\varepsilon^{-1}, \tag{2.8}$$

since $\{f_m;\ m \in N\}$ is equicontinuous on $[0, b]$. Pick $t_i = (i/k_\varepsilon)\, b \in [0, b]$, $i = 1, ..., k_\varepsilon$. Then there is $m_\varepsilon \in N$ such that

$$\|f_m(t_i) - f_l(t_i)\| < \frac{\varepsilon}{3}, \qquad m, l \geqslant m_\varepsilon,\ i = 1, ..., k_\varepsilon. \tag{2.9}$$

Combining (2.8) and (2.9) gives

$$\|f_m(t) - f_l(t)\| < \varepsilon, \qquad m, l \geqslant m_\varepsilon,\ t \in [0, b].$$

Consequently, $f_m(t)$ converges as $m \to \infty$ uniformly on $[0, b]$.

(ii) ⇒ (i). Fix $t \in [0, \infty)$. Then for each $\varepsilon > 0$, there exists $m_0 \in N$ such that

$$\|f_m(s) - f_{m_0}(s)\| < \frac{\varepsilon}{3}, \qquad m \geqslant m_0,\ s \in [0, t+1].$$

On the other hand, there exists $\delta_0 > 0$ such that

$$\|f_m(s) - f_m(t)\| < \frac{\varepsilon}{3}, \qquad m = 1, 2, ..., m_0,\ s \in [0, t+1] \text{ with } |s - t| < \delta_0, \tag{2.10}$$

since f_m is continuous on $[0, t+1]$. Hence for $m > m_0$ and s as in (2.10),

$$\|f_m(s) - f_m(t)\| \leqslant \|f_m(s) - f_{m_0}(s)\| + \|f_{m_0}(s) - f_{m_0}(t)\| + \|f_{m_0}(t) - f_m(t)\|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \tag{2.11}$$

In conclusion, $\{f_m;\ m \in N\}$ is equicontinuous at t. By using (1.3), (2.2), and the dominated convergence theorem, we obtain the existence of $\lim_{m \to \infty} F_m(\lambda)$ for $\lambda > \omega + 1$. The proof is then complete.

For the case as in Theorem 2.2(i), (2.7) can be obtained by a direct application of the vector valued version of the Post–Widder inversion formula for Laplace transforms, since $t \to \mathbf{f}(t) = (f_m(t))_{m \in N}$ is a continuous $\mathbf{E}$-valued function and $\mathbf{F}(\lambda) = \int_0^\infty e^{-\lambda t} \mathbf{f}(t)\, dt$, $\lambda > \omega + 1$.

The following example shows that without the equicontinuity of f_m in Theorem 2.2(i), the existence of $\lim_{m \to \infty} F_m(\lambda)$ alone does not imply the convergence of $f_m(t)$, even for a single t and even in the scalar case.

Example 2.3. Let $E = R$ and $f_m(t) = \cos(mt + m)$, $m \in N$. Then $f_m \in C([0, \infty), E)$ and $\|f_m(t)\| \leqslant 1$, $t \in [0, \infty)$. We see that for $\lambda > 0$,

$$F_m(\lambda) = \int_0^\infty e^{-\lambda t} \cos(mt + m)\, dt = \frac{1}{\lambda^2 + m^2} (\lambda \cos m - m \sin m)$$
$$\to 0, \qquad \text{as} \quad m \to \infty.$$

But for any $t \in [0, \infty)$, $f_m(t)$ does not converge.

Corollary 2.4. *Let $\gamma \in (0, 1]$. For each $m \in N$, let $f_m \colon [0, \infty) \to E$ satisfy* $\sup_{m \in N} \|f_m(0)\| < \infty$ *and* $\|f_m(t+h) - f_m(t)\| \leqslant M e^{\omega(t+h)} h^\gamma$ $(t, h \geqslant 0)$, *and let F_m be as in* (2.2). *Then the following statements are equivalent.*

(i) $\lim_{m \to \infty} F_m(\lambda)$ *exists for* $\lambda > \omega$.

(ii) *There exists $f \colon [0, \infty) \to E$ with $\|f(t+h) - f(t)\| \leqslant M e^{\omega(t+h)} h^\gamma$ $(t, h \geqslant 0)$ such that* $\lim_{m \to \infty} F_m(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$ *uniformly on* $[\omega + \sigma, \infty)$ *for any* $\sigma > 0$.

(iii) $\lim_{m \to \infty} f_m(t)$ *exists for all* $t \geqslant 0$.

(iv) *There exists $f \colon [0, \infty) \to E$ with $\|f(t+h) - f(t)\| \leqslant M e^{\omega(t+h)} h^\gamma$ $(t, h \geqslant 0)$ such that* $\lim_{m \to \infty} f_m(t) = f(t)$ *uniformly on bounded intervals of* $t \geqslant 0$.

Proof. From the hypothesis we know that $\|f_m(t)\| \leqslant M_1 e^{(\omega+1)t}$ $(t \geqslant 0, m \in N)$ for some constant $M_1 > 0$, and $\{f_m;\ m \in N\}$ is equicontinuous at each point $t \in [0, \infty)$. Accordingly, the equivalence of (i), (iii), and (iv) follows from Theorem 2.2.

(iv) ⇒ (ii). It is easily deduced by observing that for each fixed $\sigma > 0$

$$\left\| F_m(\lambda) - \int_0^\infty e^{-\lambda t} f(t)\, dt \right\|$$

$$\leqslant \int_0^\infty e^{-(\omega+\sigma)t} \| f_m(t) - f(t) \|\, dt, \qquad \lambda \in [\omega+\sigma, \infty).$$

(ii) ⇒ (i). Obvious.

The proof is then complete.

3. APPROXIMATION OF INTEGRATED SEMIGROUPS

First, we recall the definition of integrated semigroups.

DEFINITION 3.1. Let $r \geqslant 0$. If $(\omega, \infty) \subset \rho(A)$ and there exists a strongly continuous family $S(\cdot): [0, \infty) \to \mathbf{L}(E)$ with $\|S(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$ such that

$$R(\lambda; A)\, u = \lambda^r \int_0^\infty e^{-\lambda t} S(t)\, u\, dt, \qquad \lambda > \omega,\ u \in E,$$

then we say that A is the generator of an r-times integrated semigroup $S(\cdot)$, and denote by $A \in G_r(M, \omega)$.

It is known that

$$S(0) = 0, \qquad S(t)\, u = \frac{t^r}{\Gamma(r+1)}\, u + \int_0^t S(s)\, Au\, ds,\ t \geqslant 0,\ u \in \mathscr{D}(A). \tag{3.1}$$

LEMMA 3.2. *Let $A_m \in G_r(M, \omega)$ $(m \in N)$. Assume that* $\lim_{m\to\infty} R(\lambda_0; A_m)u$ *exists for some $\lambda_0 > \omega$ and for all $u \in E$. Then* $\lim_{m\to\infty} R(\lambda; A_m)u$ *exists for all $\lambda > \omega$ and $u \in E$.*

Proof. Using the same type of arguments as in [5, Theorem 3.1; 7, Proposition 3.4.4; or 9, Theorem 1.7.8], we can prove this lemma.

THEOREM 3.3. *Let $A, A_m \in G_r(M, \omega)$, $m \in N$. Let $S(\cdot)$ and $S_m(\cdot)$ be the r-times integrated semigroups generated by A and A_m respectively. Then the following statements are equivalent.*

(i) *For all $u \in E$,* $\lim_{m\to\infty} R(\lambda; A_m)\, u = R(\lambda; A)\, u$ *for some/all $\lambda > \omega$, and $\{S_m(\cdot)\, u;\ m \in N\}$ is equicontinuous at each point $t \in [0, \infty)$.*

(ii) *For all $u \in E$,* $\lim_{m\to\infty} S_m(t)\, u = S(t)\, u$ *uniformly on compacts of $t \geqslant 0$.*

Proof. Apply Theorem 2.2 with $f_m(t) = S_m(t)\, u$ $(m \in N)$ and $F_m(\lambda) = \lambda^{-r} R(\lambda; A_m)\, u$.

COROLLARY 3.4. *Suppose that $S(\cdot)$ and $S_m(\cdot)$ are r-times integrated semigroups on E with generators A and A_m respectively, satisfying*

$$\|S_m(t+h) - S_m(t)\| \leqslant M e^{\omega(t+h)} h^{\gamma}, \qquad m \in N,\ t, h \geqslant 0, \tag{3.2}$$

for some $\gamma \in (0, 1]$. Then the following assertions are equivalent.

(i) $\lim_{m \to \infty} R(\lambda; A_m)\, u = R(\lambda; A)\, u$ *for all* $u \in E$ *and for some/all* $\lambda > \omega$.

(ii) $\lim_{m \to \infty} S_m(t)\, u = S(t)\, u$ *for all* $u \in E$ *and uniformly on compacts of* $t \geqslant 0$.

COROLLARY 3.5. *Let A, A_m, $S(\cdot)$ and $S_m(\cdot)$ be as in Theorem* 3.3. *Let D be a core for A. Then the implications*

$$\text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iii)}$$

hold among the following statements.

(i) $\lim_{m \to \infty} R(\lambda; A_m)\, u = R(\lambda; A)\, u$ *for all* $u \in E$ *and for some/all* $\lambda > \omega$.

(ii) *For each $u \in D$, there exists $u_m \in \mathscr{D}(A_m)$ $(m \in N)$ such that* $\lim_{m \to \infty} u_m = u$ *and* $\lim_{m \to \infty} A_m u_m = Au$.

(iii) $\lim_{m \to \infty} S_m(t)\, u = S(t)\, u$ *for all* $u \in \overline{\mathscr{D}(A)}$ *and uniformly on compacts of* $t \geqslant 0$.

When $\mathscr{D}(A)$ is dense in E, the statements (i), (ii), *and* (iii) *are equivalent.*

Proof. The equivalence of (i) and (ii) is known (cf. [7, Theorem 3.4.8; 17, Theorem 2.3]).

(i) $\Rightarrow$ (iii). Fix $u \in \mathscr{D}(A)$ and put $v = (\lambda - A)\, u$. Observe that for $t \geqslant 0$, $m \in N$,

$$\begin{aligned} S_m(t)\, u &= S_m(t)\, R(\lambda; A)\, v \\ &= S_m(t)(R(\lambda; A)\, v - R(\lambda; A_m)\, v) + \frac{t^r}{\Gamma(r+1)}\, R(\lambda; A_m)\, v \\ &\quad - \int_0^t S_m(s)\, v\, ds + \lambda \int_0^t S_m(s)\, R(\lambda; A_m)\, v\, ds \end{aligned}$$

by (3.1). Hence we deduce easily, by (i) and

$$\|S_m(t)\| \leqslant Me^{\omega t} \qquad (t \geqslant 0,\ m \in N), \tag{3.3}$$

that $\{S_m(\cdot)\, u;\ m \in N\}$ is equicontinuous at each point $t \in [0, \infty)$. With the help of Theorem 2.2 we obtain that (iii) holds true for all $u \in \mathscr{D}(A)$, and so for all $u \in \overline{\mathscr{D}(A)}$ owing to (3.3).

When $\mathscr{D}(A)$ is dense in E, $\overline{\mathscr{D}(A)} = E$. Therefore (iii) implies (i) in view of Theorem 3.3.

THEOREM 3.6. *For each $m \in N$, let $A_m \in G_r(M, \omega)$ and $S_m(\cdot)$ be the r-times integrated semigroup generated by A_m, such that for every $u \in E$ and $t \geqslant 0$, $\{S_m(\cdot)\, u;\ m \in N\}$ is equicontinuous at t. Let $\lambda_0 > \omega$. Assume that for every $u \in E$, the limit*

$$R(\lambda_0)\, u := \lim_{m \to \infty} R(\lambda_0; A_m)\, u \textit{ exists}, \qquad \textit{with } \mathscr{N}(R(\lambda_0)) = 0. \tag{3.4}$$

Then there exists $A \in G_r(M, \omega)$ generating an r-times integrated semigroup $S(\cdot)$ such that $\lim_{m \to \infty} S_m(t)\, u = S(t)\, u$ for all $u \in E$ and uniformly on compacts of $t \geqslant 0$.

Proof. By Definition 3.1 we have

$$\lambda^{-r} R(\lambda; A_m)\, u = \int_0^\infty e^{-\lambda t} S_m(t)\, u\, dt, \qquad \lambda > \omega,\ u \in E,\ m \in N. \tag{3.5}$$

Moreover, we know by Lemma 3.2 that for any $u \in E$, $R(\lambda)\, u := \lim_{m \to \infty} R(\lambda; A_m)\, u$ exists for all $\lambda > \omega$. So an appeal to Theorem 2.2 yields that for each $u \in E$, $S(t)\, u := \lim_{m \to \infty} S_m(t)\, u$ exists uniformly on compacts of $t \geqslant 0$.

It is clear that $R(\lambda)$ is a psendo resolvent on $\lambda > \omega$. Define

$$A = \lambda_0 I - R(\lambda_0)^{-1}.$$

It can be seen that $R(\lambda) = R(\lambda; A)$, $\lambda > \omega$. Consequently, we obtain by (3.5) that for $\lambda > \omega$ and $u \in E$,

$$\lambda^{-r} R(\lambda; A)\, u = \int_0^\infty e^{-\lambda t} S(t)\, u\, dt. \tag{3.6}$$

This justifies the required conclusion.

COROLLARY 3.7. *Suppose that $S_m(\cdot)$, $m \in N$, are r-times integrated semigroups on E with generators A_m, satisfying* (3.2). *If* (3.4) *holds, then the conclusion of Theorem* 3.6 *holds.*

COROLLARY 3.8. *For each $m \in N$, let $A_m \in G_r(M, \omega)$ with (3.4) holding, and let $S_m(\cdot)$ be the r-times integrated semigroup generated by A_m. Then there is a closed linear operator A such that the part of A in $\overline{\mathscr{D}(A)}$ generates an r-times integrated semigroup $S(\cdot)$ on $\overline{\mathscr{D}(A)}$ and*

$$\lim_{m \to \infty} S_m(t)\, u = S(t)\, u, \quad u \in \overline{\mathscr{D}(A)}$$

uniformly on compacts of $t \geqslant 0$.

If $\mathscr{R}(R(\lambda_0))$ is dense in E, in addition, then the conclusion of Theorem 3.6 holds.

Proof. We know, from the proofs of Corollary 3.5 and Theorem 3.6, that there exists A with $\lim_{m \to \infty} R(\lambda; A_m)\, u = R(\lambda; A)\, u$, $\lambda > \omega$, and that for each $u \in \mathscr{D}(A)$ and $t \geqslant 0$, $\{S_m(\cdot)\, u; m \in N\}$ is equicontinuous at t. Accordingly, the limit

$$S(t)\, u := \lim_{m \to \infty} S_m(t)\, u \quad (u \in \overline{\mathscr{D}(A)})$$

exists uniformly on compacts of $t \geqslant 0$, in view of Theorem 2.2. Therefore (3.6) holds for $\lambda > \omega$ and $u \in \overline{\mathscr{D}(A)}$, which implies the first conclusion.

The density of $\mathscr{R}(R(\lambda_0))$ implies the density of A. Thus the second conclusion follows from the first one.

Remark 3.9. For $\gamma = 1$ and $r = 1$, Corollary 3.4 and Corollary 3.7 coincide with Theorem 3.3 and Corollary 3.5 in [5]. In the case of $r \in N_0$, Corollary 3.5 and Corollary 3.8 can be found in [17, 18].

4. APPROXIMATION OF SOLUTIONS OF NONHOMOGENEOUS CAUCHY PROBLEMS

Let $x \in E$ and $f \in C([0, b], E)$ (where $b > 0$). We consider the nonhomogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, b], \\ u(0) = x. \end{cases} \tag{4.1}$$

By a mild solution of (4.1), we understand a function $u(\cdot) \in C([0, b], E)$ satisfying that $\int_0^t u(s)\, ds \in \mathscr{D}(A)$ for all $t \in [0, b]$ and

$$u(t) = x + A \int_0^t u(s)\, ds + \int_0^t f(s)\, ds, \qquad t \in [0, b].$$

In the sequel, we say that $A \in H_r(M, \omega)$ if $(\omega, \infty) \subset \rho(A)$ and

$$\|(\lambda^{-r}R(\lambda; A))^{(m)}\| \leqslant Mm!(\lambda-\omega)^{-m-1}, \qquad \lambda > \omega,\ m \in N_0.$$

LEMMA 4.1. *Assume that* $A \in H_r(M, \omega)$, $f(0) \in \mathscr{D}(A^{[r]})$, *and*

$$f(t) = f(0) + (j_{r-1} * g)(t), \qquad t \in [0, b]$$

for some

$$g \in \begin{cases} C([0, b], E) & \text{if} \quad r \in [0, 1), \\ L^1([0, b], E) & \text{otherwise.} \end{cases}$$

Let $x \in \mathscr{D}(A^{[r]})$ *with*

$$A^{[r]}x \in \begin{cases} \overline{\mathscr{D}(A)} & \text{if} \quad r \in N_0 \\ \mathscr{D}(A) & \text{if} \quad r \notin N_0. \end{cases}$$

Then (4.1) *admits a unique mild solution, given by*

$$u(t) = \frac{d}{dt}(S(t)\, A^{[r]}x) + S(t)\, A^{[r]}f(0)$$

$$+ \frac{d}{dt}(S * j_{r-[r]-1} * g)(t) + h(t), \qquad t \in [0, b],$$

where $S(\cdot)$ *is the* $([r]+1)$-*times integrated semigroup generated by* A, *and*

$$h(t) := \begin{cases} 0 & \text{if} \quad r \in [0, 1), \\ \displaystyle\sum_{j=0}^{[r]-1} \left(\frac{t^{j+1}}{(j+1)!} A^j f(0) + \frac{t^j}{j!} A^j x \right) & \text{otherwise.} \end{cases}$$

Proof. Proceed analogously as in the proof of [12, Theorem 4.6].

The following characterization (cf. [12; 24, Chap. 1]) of the generators of integrated semigroups will be used freely in the proof of Theorem 4.2 and Example 5.1.

Let $\bar{r} \in (r, r+1]$. *Then* A *is the generator of an* $\bar{r}$-*times integrated semigroup* $S(\cdot)$ *satisfying*

$$\|(S * j_{r-\bar{r}})(t+h) - (S * j_{r-\bar{r}})(t)\| \leqslant Me^{\omega(t+h)}h, \qquad t, h \geqslant 0,$$

if and only if $A \in H_r(M, \omega)$. *In this case, we also have*

$$\|S(t+h) - S(t)\| \leqslant \frac{2M}{(\bar{r}-r)\,\Gamma(\bar{r}-r)}\, e^{\omega(t+h)}h^{\bar{r}-r}, \qquad t, h \geqslant 0.$$

THEOREM 4.2. *Let the hypotheses of Lemma* 4.1 *hold for* A, f, g, x, *and also for* A_m, f_m, g_m, x_m $(m \in N)$ *in place of* A, f, g, x, *respectively. Suppose*

(i) $\lim_{m\to\infty} R(\lambda; A_m)\, v = R(\lambda; A)\, v$ *for all* $v \in E$ *and for some* $\lambda > \omega$.

(ii) $\lim_{m\to\infty} A_m^i f_m(0) = A^i f(0)$, $0 \leqslant i \leqslant [r]$, *and* $\lim_{m\to\infty} \|g_m - g\|_{L^1([0,b], E)} = 0$.

(iii) $\lim_{m\to\infty} A_m^i x_m = A^i x$ $(0 \leqslant i \leqslant r)$ *and* $\overline{\mathscr{D}(A_m)} \supset \mathscr{D}(A)$ $(m \in N)$, *if* $r \in N_0$; $\lim_{m\to\infty} A_m^i x_m = A^i x$ $(0 \leqslant i \leqslant [r]+1)$, *if* $r \notin N_0$.

Then the mild solution $u_m(t)$ *of*

$$\begin{cases} u_m'(t) = A_m u_m(t) + f_m(t), & t \in [0, b] \\ u_m(0) = x_m \end{cases} \tag{4.2}$$

converges to the mild solution $u(t)$ *of* (4.1) *uniformly for* $t \in [0, b]$.

Proof. We denote by $S_m(\cdot)$, $m \in N$, the $([r]+1)$-times integrated semigroups generated by A_m. For convenience, write

$$A_0 = A, \qquad S_0 = S, \qquad f_0 = f, \qquad g_0 = g, \qquad x_0 = x.$$

We have that for any $m \in N_0$ and $t, h \geqslant 0$,

$$\|S_m(t+h) - S_m(t)\| \leqslant M_1\, e^{\omega(t+h)} h^{[r]-r+1}, \tag{4.3}$$

$$\|\tilde{S}_m(t+h) - \tilde{S}_m(t)\| \leqslant M e^{\omega(t+h)} h, \tag{4.4}$$

where M_1 is a constant and $\tilde{S}_m := S_m * j_{r-[r]-1}$. This, together with condition (i), yields by Corollary 3.4 that for each $v \in E$,

$$\lim_{m\to\infty} S_m(t)\, v = S_0(t)\, v \text{ uniformly on compacts of } t \geqslant 0, \tag{4.5}$$

$$\lim_{m\to\infty} \tilde{S}_m(t)\, v = \tilde{S}_0(t)\, v \text{ uniformly on compacts of } t \geqslant 0. \tag{4.6}$$

Moreover, it follows from (4.6) that, given $\phi \in C([0, b], E)$,

$$\lim_{m\to\infty} (\tilde{S}_m * \phi)(t) = (\tilde{S}_0 * \phi)(t), \qquad t \in [0, b]. \tag{4.7}$$

We observe by (4.4) that $\{\tilde{S}_m * \phi;\, m \in N\}$ is equicontinuous on $[0, b]$, and so conclude (cf. the proof of Theorem 2.2) that the convergence in (4.7) is uniform for all t in $[0, b]$.

Lemma 4.1 says that for every $m \in N_0$ and $t \geqslant 0$,

$$u_m(t) = \frac{d}{dt}\,(S_m(t)\, A_m^{[r]} x_m) + \frac{d}{dt}\,(\tilde{S}_m * g_m)(t)$$
$$+ S_m(t)\, A_m^{[r]} f_m(0) + h_m(t), \tag{4.8}$$

where

$$h_m(t) := \begin{cases} 0 & \text{if} \quad r \in [0, 1), \\ \sum_{j=0}^{[r]-1} \left(\frac{t^{j+1}}{(j+1)!} A_m^j f_m(0) + \frac{t^j}{j!} A_m^j x_m \right) & \text{otherwise.} \end{cases}$$

It is immediate, by (4.5), conditions (ii) and (iii), that

$$\lim_{m \to \infty} S_m(t) A_m^{[r]} f_m(0) = S_0(t) A_0^{[r]} f_0(0), \qquad \lim_{m \to \infty} h_m(t) = h_0(t), \tag{4.9}$$

uniformly on the t-interval $[0, b]$.

Next, we show that

$$\lim_{m \to \infty} \frac{d}{dt} (\tilde{S}_m * g_m)(t) = \frac{d}{dt} (\tilde{S}_0 * g_0)(t), \qquad \text{uniformly for } t \in [0, b]. \tag{4.10}$$

To this end, we define the linear mappings Γ_m $(m \in N_0)$: $L^1([0, b], E) \to C([0, b], E)$ by

$$(\Gamma_m \phi)(t) = \frac{d}{dt} (\tilde{S}_m * \phi)(t), \qquad t \in [0, b], \ \phi \in L^1([0, b], E).$$

Γ_m are well defined by (4.4) and the density of $C^1([0, b], E)$ in $L^1([0, b], E)$. Observing by (4.4) that for any $m \in N_0$,

$$\begin{aligned} \|\Gamma_m \phi\|_{C([0, b], E)} &= \max_{t \in [0, b]} \left\| \lim_{h \to 0^+} \int_0^t h^{-1} (\tilde{S}_m(t-s+h) - \tilde{S}_m(t-s)) \, \phi(s) \, ds \right\| \\ &\leqslant M e^{\omega b} \|\phi\|_{L^1([0, b], E)}, \qquad \phi \in L^1([0, b], E), \end{aligned}$$

we see that

$$\|\Gamma_m\|_{L^1 \to C} \leqslant M e^{\omega b}, \qquad m \in N_0. \tag{4.11}$$

Using (4.7) and the statement below it yields that if $\phi \in C^1([0, b], E)$,

$$\begin{aligned} \|\Gamma_m \phi - \Gamma_0 \phi\|_{C([0, b], E)} &= \max_{t \in [0, b]} \|[(\tilde{S}_m - \tilde{S}_0) * \phi'](t)\| \\ &\to 0 \qquad \text{as} \quad m \to \infty. \end{aligned}$$

In combination with (4.11), this implies that for all $\phi \in L^1([0, b], E)$,

$$\|\Gamma_m \phi - \Gamma_0 \phi\|_{C([0, b], E)} \to 0 \qquad \text{as} \quad m \to \infty,$$

since $C^1([0, b], E)$ is dense in $L^1([0, b], E)$. Accordingly,

$$\begin{aligned}\|\Gamma_m g_m - \Gamma_0 g_0\|_{C([0,b],E)} &\leqslant \|\Gamma_m\|_{L^1 \to C} \|g_m - g_0\|_{L^1([0,b],E)} \\ &\quad + \|\Gamma_m g_0 - \Gamma_0 g_0\|_{C([0,b],E)} \\ &\to 0 \qquad \text{as} \quad m \to \infty,\end{aligned}$$

by condition (ii). Therefore (4.10) holds.

It remains to show that

$$\lim_{m\to\infty} \frac{d}{dt}(S_m(t)\, A_m^{[r]} x_m) = \frac{d}{dt}(S_0(t)\, A_0^{[r]} x_0), \qquad \text{uniformly for} \quad t \in [0, b]. \tag{4.12}$$

If $r \notin N_0$, then $x_m \in \mathscr{D}(A_m^{[r]+1})$ $(m \in N_0)$ by hypothesis, so that (4.12) follows from (4.5) and condition (iii) immediately, because

$$\frac{d}{dt}(S_m(t)\, A_m^{[r]} x_m) = \frac{t^{[r]}}{[r]!} A_m^{[r]} x_m + S_m(t)\, A_m^{[r]+1} x_m, \qquad t \in [0, b],\ m \in N_0,$$

by (3.1). Let now $r \in N_0$. We have by hypothesis that for any $m \in N_0$,

$$A_m^r x_m \in \overline{\mathscr{D}(A_m)}, \qquad \lim_{m\to\infty} A_m^r x_m = A_0^r x_0, \tag{4.13}$$

and

$$\overline{\mathscr{D}(A_m)} \supset \mathscr{D}(A_0). \tag{4.14}$$

Consider the linear mappings $\Lambda_m (m \in N_0)$: $\overline{\mathscr{D}(A_m)} \to C([0, b], E)$, given by

$$(\Lambda_m v)(t) = \frac{d}{dt}(S_m(t)\, v), \qquad t \in [0, b],\ v \in \overline{\mathscr{D}(A_m)}.$$

From (4.3) and noting $[r] - r + 1 = 1$, we obtain that for $m \in N_0$ and $v \in \overline{\mathscr{D}(A_m)}$,

$$\begin{aligned}\|\Lambda_m v\|_{C([0,b],E)} &= \max_{t\in[0,b]} \left\| \lim_{h\to 0^+} [h^{-1}(S_m(t+h) - S_m(t))\, v] \right\| \\ &\leqslant M_1\, e^{\omega b} \|v\|;\end{aligned}$$

hence

$$\|\Lambda_m\| \leqslant M_1 e^{\omega t}, \qquad m \in N_0. \tag{4.15}$$

We fix $v \in \mathscr{D}(A_0)$. The condition (i) gives, by Corollary 3.5, that there exist $v_m \in \mathscr{D}(A_m)$ such that $v_m \to v$ and $A_0 v_m \to A_0 v$. On the other hand, we have by (3.1) that

$$(\Lambda_m v_m)(t) = \frac{t^r}{r!}\, v_m + S_m(t)\, A_m v_m\,, \qquad t \in [0, b],\ m \in N_0\,.$$

So using (4.5) gives

$$\lim_{m \to \infty} \|\Lambda_m v_m - \Lambda_0 v\|_{C([0, b], E)} = 0.$$

It follows from (4.14) and (4.15) that

$$\begin{aligned}\|\Lambda_m v - \Lambda_0 v\|_{C([0, b], E)} &\leqslant \|\Lambda_m\|\ \|v_m - v\| + \|\Lambda_m v_m - \Lambda_0 v\|_{C([0, b], E)}\\ &\to 0 \qquad \text{as} \quad m \to \infty,\end{aligned}$$

which implies that for all $v \in \overline{\mathscr{D}(A_0)}$,

$$\lim_{m \to \infty} \|\Lambda_m v - \Lambda_0 v\|_{C([0, b], E)} = 0.$$

Consequently

$$\begin{aligned}\|\Lambda_m(A_m^r x_m) - \Lambda_0(A_0^r x_0)\|_{C([0, b], E)} &\leqslant \|\Lambda_m\|\ \|A_m^r x_m - A_0^r x_0\|\\ &\quad + \|\Lambda_m(A_0^r x_0) - \Lambda_0(A_0^r x_0)\|_{C([0, b], E)}\\ &\to 0 \qquad \text{as} \quad m \to \infty,\end{aligned}$$

by (4.13) and (4.15). This ends the proof.

COROLLARY 4.3. *Let $0 < r < 1$. For each $m \in N_0$, let $A_m \in H_r(M, \omega)$, $x_m \in \mathscr{D}(A_m)$ and*

$$f_m(t) = f_m(0) + (j_{r-1} * g_m)(t), \qquad t \in [0, b],$$

where $g_m \in C([0, b], E)$, $A_0 := A$, $x_0 := x$, $f_0 := f$, and $g_0 := g$. Suppose

(i) $\lim_{m \to \infty} R(\lambda; A_m)\, v = R(\lambda; A)\, v$ *for all $v \in E$ and for some $\lambda > \omega$.*

(ii) $\lim_{m \to \infty} f_m(0) = f(0)$ *and* $\lim_{m \to \infty} \|g_m - g\|_{L^1([0, b], E)} = 0$.

(iii) $\lim_{m \to \infty} x_m = x$ *and* $\lim_{m \to \infty} A_m x_m = Ax$.

Then the conclusion of Theorem 4.2 holds.

COROLLARY 4.4. *For each* $m \in N_0$, *let* $A_m \in H_0(M, \omega)$, $x_m \in \overline{\mathscr{D}(A_m)}$ *and* $f_m \in C([0, b], E)$, *where* $A_0 := A$, $x_0 := x$, *and* $f_0 := f$. *Suppose*

(i) $\lim_{m \to \infty} R(\lambda; A_m) v = R(\lambda; A) v$ *for all* $v \in E$ *and for some* $\lambda > \omega$.

(ii) $\lim_{m \to \infty} \|f_m - f\|_{L^1[0, b], E)} = 0$.

(iii) $\lim_{m \to \infty} x_m = x$ and $\overline{\mathscr{D}(A_m)} \supset \mathscr{D}(A)$ $(m \in N)$.

Then the conclusion of Theorem 4.2 *holds.*

Proof. Note that, in this case, the term $\frac{d}{dt}(\tilde{S}_m * g_m)(t) + S_m(t) A_m^{[r]} f_m(0)$ in (4.8) can be rewritten as $\frac{d}{dt}(\tilde{S}_m * f_m)(t)$.

Remark 4.5. Specialized to the case of $f_m = f$ and $x_m = x$ for all $m \in N$, Corollary 4.4 improves Theorem 4.3 in [5], by replacing the condition that $\mathscr{D}(A_m) \supset \mathscr{D}(A)$ and $S_m(t)$: $\mathscr{D}(A) \to \overline{\mathscr{D}(A)}$, by $\overline{\mathscr{D}(A_m)} \supset \mathscr{D}(A)$.

5. EXAMPLES

EXAMPLE 5.1. We consider the linear operators $A_m (m \in N_0)$ with the property that for all $m \in N_0$,

$$\rho(A_m) \supset \{\lambda \in \mathbf{C};\ \operatorname{Re} \lambda > \omega\},$$

$$\|R(\lambda; A_m)\| \leqslant C|\lambda|^{\mu}, \qquad \operatorname{Re} \lambda > \omega,$$

where $C, \omega > 0$, $\mu > -1$ are constants. It is known from [2] (see also [24, Chap. 1]) that to each $\alpha > 1$, there corresponds a $C_\alpha > 0$ such that $A_m \in G_{\mu + \alpha}(C_\alpha, \omega)$ for all $m \in N_0$. Now fix $\alpha > 1$ and denote by $S_m(\cdot)$ the $(\mu + \alpha)$-times integrated semigroups generated by A_m. Letting $\gamma = \min\{1, \frac{1}{2}(\alpha - 1)\}$, we see that (3.2) is satisfied, since $A_m \in H_{\mu + \alpha - \gamma}(C_{\alpha - \gamma}, \omega)$. Therefore we conclude by Corollary 3.4 that $S_m(t)$ converges strongly to $S_0(t)$ uniformly on compacts of $t \geqslant 0$, provided $R(\lambda; A_m)$ converges strongly to $R(\lambda; A_0)$ for some $\lambda > \omega$.

EXAMPLE 5.2. Let $\{a_{ij, m}\}_{m \in N_0} \subset W^{3, \infty}(R^n, R)$, $\{b_{j,m}\}_{m \in N_0} \subset W^{1, \infty}(R^n, \mathbf{C})$ for all $i, j \in \{1, ..., n\}$, and $\{c_m\}_{m \in N_0} \subset L^\infty(R^n, \mathbf{C})$, satisfying that for $i, j \in \{1, ..., n\}$,

$$\sup_{m \in N} (\|a_{ij, m}\|_{W^{1, \infty}} + \|b_{j, m}\|_{L^\infty} + \|c_m\|_{L^\infty}) < \infty, \tag{5.1}$$

$$\lim_{m \to \infty} a_{ij, m}(y) = a_{ij, 0}(y), \quad \lim_{m \to \infty} b_{j, m}(y) = b_{j, 0}(y),$$

$$\lim_{m \to \infty} c_m(y) = c_0(y), \qquad y \in R^n. \tag{5.2}$$

Suppose there exists a $\mu_0 > 0$ such that

$$\sum_{i,\, j=1}^{n} a_{ij,\, m}(y)\, \xi_i \xi_j \geqslant \mu_0\, |\xi|^2, \qquad y, \xi \in R^n,\ m \in N_0. \tag{5.3}$$

We now consider the quadratic forms $Q_m\colon H^1(R^n) \times H^1(R^n) \to \mathbf{C}$ defined by

$$Q_m(u, v) = \sum_{i,\, j=1}^{n} \int_{R^n} \tilde{a}_{ij,\, m} D_i u\, \overline{D_j v}\, dy + \sum_{j=1}^{n} \int_{R^n} \tilde{b}_{j,\, m} D_j u \bar{v}\, dy + \int_{R^n} c_m u \bar{v}\, dy,$$

where for $i, j \in \{1, ..., n\}$, $m \in N_0$,

$$\tilde{a}_{ij,\, m} := \frac{1}{2}\,(a_{ij,\, m} + a_{ji,\, m}), \qquad \tilde{b}_{j,\, m} := b_{j,\, m} - \sum_{i=1}^{n} D_i \tilde{a}_{ij,\, m}.$$

For each $m \in N_0$, let $-A_m$ be the operator associated with Q_m, and T_m the semigroup on $L^2(R^n)$ generated by A_m. We obtain, in view of Theorems 3.1 and 5.3 of [3] and their proofs, that for any $m \in N_0$, T_m interpolates to provide a C_0 semigroup $T_{p,\, m}$ on each $L^p(R^n)$ for $1 \leqslant p < \infty$, that $T_{p,\, m}$ is holomorphic of angle $\frac{\pi}{2}$, and that the corresponding integral kernel $K_m(z;\, y, \eta)$ satisfies

$$|K_m(z;\, y, \eta)| \leqslant C e^{\omega\, \mathrm{Re}\, z} (\mathrm{Re} z)^{-n/2}\, e^{-b\, |y-\eta|^2\, |z|^{-1}}, \quad \mathrm{Re}\, z > 0,\ y, \eta \in R^n,\ m \in N_0,$$

for certain constants C, b, $\omega > 0$, independent of m. With the help of the Riesz interpolation theorem, a standard argument shows that

$$\|T_{p,\, m}(z)\| \leqslant M e^{\omega\, \mathrm{Re}\, z} \left(\frac{|z|}{\mathrm{Re}\, z}\right)^{2n\, |1/2 - 1/p|}, \qquad \mathrm{Re}\, z > \omega,\ m \in N_0,\ 1 \leqslant p < \infty.$$

Denoting by $A_{p,\, m}$ the generators of $T_{p,\, m}$, we get by Theorem 2.3 in [6] that for any $\alpha > 2n\, |\frac{1}{2} - \frac{1}{p}|$, there is a constant $M_\alpha > 0$ such that $iA_{p,\, m} \in G_\alpha(M_\alpha, 1)$, $m \in N_0$, $1 \leqslant p < \infty$. Obviously, $iA_{p,\, m} \in H_\alpha(M_\alpha, 1)$.

By Theorem 2.7 in [21, Chap. V], we have that $C_c^\infty(R^n)$ is a core for $A_{p,\, m}$ ($1 \leqslant p < \infty$, $m \in N_0$), and it is clear that

$$A_{p,\, m} u = \sum_{i,\, j=1}^{n} a_{ij,\, m} D_i D_j u - \sum_{j=1}^{n} b_{j,\, m} D_j u - c_m u, \quad u \in W^{2,\, p}(R^n) \subset \mathscr{D}(A_{p,\, m}).$$

Thus (5.1) and (5.2) impliy

$$\lim_{m \to \infty} A_{p,\, m} u = A_{p,\, 0} u, \qquad u \in C_c^\infty(R^n).$$

We deduce by Corollary 3.5 that for some $\lambda > \omega$,

$$\lim_{m \to \infty} R(\lambda; A_{p,m})\, u = R(\lambda; A_{p,0})\, u, \qquad \text{for all} \quad u \in L^p(R^n).$$

Consequently, we obtain, making use of Theorem 4.2, that given $x \in W^{2[2n\,|1/2 - 1/p|] + 2,\, p}(R^n)$ and f as in Lemma 4.1,

$$\lim_{m \to \infty} \|u_m(t) - u_0(t)\|_{L^p(R^n)} = 0$$

with uniform convergence for $t \in [0, b]$, where for each $m \in N_0$, $u_m(t)$ is the mild solution of

$$\begin{cases} u'_m(t) = iA_{p,m} u_m(t) + f(t), & t \in [0, b], \\ u_m(0) = x. \end{cases}$$

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